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**LETTER TO THE EDITOR**

# Field theory of the random flux model

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**Abstract.** The long-range properties of the random flux model (lattice fermions hopping under the influence of maximally random *link* disorder) are shown to be described by a supersymmetric field theory of non-linear $\sigma$-model type, where the group $GL(n|n)$ is the global invariant manifold. An extension to non-Abelian generalizations of this model identifies connections to lattice QCD, Dirac fermions in a random gauge potential, and stochastic non-Hermitian operators.

Quantum disordered systems are typically realized in Hamiltonians of the general form $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ models the underlying 'clean' system, and disorder is introduced via the randomly distributed *Hermitian* operator $\hat{V}$. Sometimes, however, it is preferable to implement disorder in terms of *unitary* stochastic operators and to consider Hamiltonians of the type

$$\hat{H} = -\sum_{\langle ij\rangle} c_i^\dagger U_{ij} c_j \tag{1}$$

where $\langle ij\rangle$ denote neighbouring sites of a $d$-dimensional hypercubic lattice, the $c$ represent $N$-component lattice fermions, and $U_{ij}$ represent $N$-dimensional unitary matrices residing on the links of the lattice. Stochasticity is introduced by drawing the $U$ from a random distribution (albeit subject to the hermiticity requirement $U_{ij} = U_{ji}^\dagger$). Hamiltonians of the type (1) are commonly referred to as random flux (RF) models, a denotation we will also adopt for the cases $N \neq 1$.

RF models appear in a variety of different contexts. The $2d$ $N = 1$ version describes the dynamics of lattice fermions subject to a random magnetic field or, more accurately, a random vector potential [1–5]. This model has been discussed in connection with the physics of the half-filled fractional quantum Hall phase [6], the physics of the spin–split Landau level [3, 7], and gauge theory of high $T_c$ superconductivity [8]. Identifying the two fermion components of the $N = 2$ RF model with a spin degree of freedom, (1) describes the propagation of lattice electrons on a spin-disordered background, a situation that occurs, e.g., in connection with the physics of manganese oxides [9]. Identifying the three fermion components of the $N = 3$ model with a colour degree of freedom, (1) represents a prototype‖ of the strong coupling lattice QCD Hamiltonian.

‖ To obtain the strong coupling QCD Hamiltonian, equation (1) must be upgraded to accommodate lattice Dirac fermions. While this additional structure can be implemented within the present analysis [30], for the purposes of this letter, we will not consider it further.

Superficially, (1) appears to fall into the general class of (bond disordered) Anderson Hamiltonians. That conjecture indeed holds true provided one stays away from the middle of the tight-binding band, $\epsilon = 0$. Upon approaching $\epsilon = 0$, however, the phenomenology of the RF models begins to differ drastically from the one of conventional disordered fermion systems. In spite of intensive numerical and analytical investigation [1–5, 7], central aspects of these deviations are not yet fully understood. For example, the key question of whether or not the $2d$ RF model possesses a band centre extended metallic phase has not yet been settled; apart from the fact that the average density of states (DoS) diverges upon approaching $\epsilon = 0$, much of the structure of even that basic observable remains unknown.

The purpose of this letter is twofold: firstly we wish to reveal a diverse network of interconnections that exist between the RF problem and related areas of current research interest. Secondly, in doing so, we provide new information regarding the band centre behaviour of the RF model.

Both aspects of that programme are based on the result that the long-range behaviour of average $n$-point Green functions, $\langle G^{\pm}(\epsilon_1)\ldots G^{\pm}(\epsilon_n)\rangle$, of the RF model can be obtained from a supersymmetric field theory defined by the action

$$S[T] = -\int [c_1 \mathrm{str}(\partial T^{-1}\partial T) + \mathrm{i}c_2 \mathrm{str}(\hat{\epsilon}(T + T^{-1})) + c_3(\mathrm{str}(T^{-1}\partial T))^2] + S_{\mathrm{b}}[T] \qquad (2)$$

where $T \in GL(n|n)$ (the group of invertible supermatrices of dimension $2n$), 'str' is the standard supertrace, and the matrix $\hat{\epsilon} = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$. The contribution $S_{\mathrm{b}}$† represents a boundary action that depends on the values of the fields $T$ at the corner points of the lattice.

Equation (2) is derived under the assumption of maximal unitary randomness, i.e. all $U_{ij} \in U(N)$, independently distributed according to the Haar measure. In this case, the constants $c_1 = Na^{2-d}/8d$, $c_2 = N(2d-1)^{1/2}a^{-d}/4d$, $c_3 = a^{2-d}C/16d$ where $a$ represents the lattice spacing, and $C$ denotes a geometry-dependent numerical constant $O(1)$. Below we will argue that the structure of the field theory is actually disorder independent, i.e. that RF models are generally described by (2)‡, where the strength of the disorder manifests itself merely in the value of the coupling constants.

Towards the end of the paper we will outline how, starting from the 'microscopic' Hamiltonian (1), the effective description (2) is derived. However, before turning to that more technical part of the discussion, we first address the question of what kind of information can be gained from the field theory. Our main goal will be to demonstrate that the action (2) represents a quantitative implementation of the network of connections displayed in figure 1. By exploring different links, we will discuss some characteristic features of the field theory.

*Chiral random matrix theory (ChRMT).* As usual with field theories of disordered systems, the low-energy regime of (2) (energies $\epsilon < c_1/(c_2L^2)$) is governed by spatially constant field configurations $T_0 = \mathrm{const.}$,

$$S_0[T_0] = -\frac{\pi\rho_0}{2}\mathrm{str}(\hat{\epsilon}(T_0 + T_0^{-1})) + S_{\mathrm{b}}[T_0] \qquad (3)$$

where $\rho_0$ is the bulk mean DoS of the system. Correlation functions computed with respect to the first contribution to the action (3) coincide with those otherwise obtained for the chiral

† Defining $N_i$, $i = 1, \ldots, d$ as the number of sites in the $\hat{e}_i$-direction,

$$S_{\mathrm{b}}[T] = \frac{N}{2^d}\sum_{s_i=0,1}(-)^{\sum_i (N_i+1)s_i}\,\mathrm{str}\,\ln(T(s_1L_1, \ldots, s_dL_d)).$$

‡ Models where the $U_{ij}$ are subject to further constraints may behave qualitatively differently and are not encompassed by the present analysis (e.g. the time-reversal invariant counterpart of the RF model, $U_{ij} \in O(N)$).

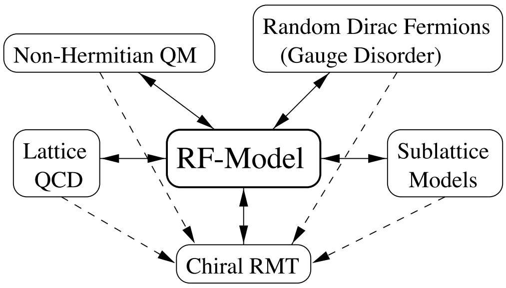


**Figure 1.** Connection between the RF model and various related systems governed by the presence of chiral symmetries. The low-energy limit of all models is universally described by ChRMT.

unitary random matrix ensemble (ChGUE) [10–13], (the ensemble of symmetry $A$III in the classification scheme of [14]), i.e. the ensemble of block off-diagonal matrices

$$\begin{pmatrix} & A \\ A^\dagger & \end{pmatrix} \tag{4}$$

where $A$ is complex random. In particular, the mean DoS is found to vanish as $\epsilon \to 0$ on a scale set by the mean level spacing. The connection to ChRMT follows readily from the fact that the RF Hamiltonian possesses a chiral symmetry: partitioned into two nested sublattices $A$ and $B$, the bipartite lattice Hamiltonian assumes a block off-diagonal form (4) in an $A/B$-decomposition. To the best of our knowledge, the ramifications of the chiral structure on the physical properties of RF models was first reported in [5]. The 'microscopic' justification of the applicability of ChRMT to the universal low-energy limit of the RF model is a new result. (In fact, and in contrast to conventional disordered systems, that limit does not behave *absolutely* universally: the fine structure of the DoS close to $\epsilon = 0$ depends on the 'parity' of the lattice, i.e. on whether the number of sites is even or odd†. Without going into details, we remark that the information about this effect is encoded in the boundary term $S_b$.)

*Non-Hermitian operators.* To investigate problems involving non-Hermitian stochastic operators $A \neq A^\dagger$ one commonly introduces an operator like the one shown in (4), i.e. an *Hermitian* auxiliary operator of twice the dimension of the original problem [15–17]. Put differently, non-Hermitian Hamiltonians possess an inbuilt chiral structure implying that their low-energy universal properties coincide with those of manifestly chiral problems like the RF model. That in turn means that the basic structure of the low-energy field theory (2) of the RF model (a system with broken time-reversal invariance) should coincide with that of the (time-reversal non-invariant version of the) non-linear $\sigma$-model of non-Hermitian problems introduced in [16]. To make that connection explicit, we introduce the auxiliary matrix variable $Q = \exp(W\sigma_1/2)(\hat{s} \otimes \sigma_3)\exp(-W\sigma_1/2)$, where $W = \ln T$, $\sigma_i$ are Pauli matrices operating in the block space of (4), and $\hat{s} = \mathrm{diag}(\mathrm{sgn}\,\mathrm{Im}\,\epsilon_1, \ldots, \mathrm{sgn}\,\mathrm{Im}\,\epsilon_n)$. One may check by direct comparison that for $n = 1$ (the case there considered), the matrices $Q$ are equivalent to the degrees of freedom employed in [16]. When represented in terms of $Q$, (2) assumes a form similar to a standard [18] non-linear $\sigma$-model, albeit one of novel symmetry [16]. Indeed, this connection is not incidental, but rather extends to the more complex variants $n > 1$, and other symmetry classes [19]. (For a more thorough discussion of these symmetry aspects, we refer the reader to the original [14].)

† For an odd number of sites, the block matrices $A$ in (4) assume a *rectangular* form, the Hamiltonian matrix becomes singular and states at zero energy appear.

*Weakly disordered sublattice models.* Leaving the random matrix regime and turning to the more complicated spatially extended problem, it is important to notice that the field theory (2) has a closely related precursor. Analysis of a weakly disordered sublattice model led Gade [20] to a boson replica version of the present model, i.e. a theory over fields $T \in GL(nR)/U(nR)$, where $R \to 0$ is the number of replicas. The action for these fields coincides with (2), save for the absence of the boundary term, and the important difference that, due to the weakness of the disorder, the coupling constant $c_1$ was parametrically larger than one. However, beyond these differences, the supersymmetric extension of the theory in the present context does not change the validity of the perturbative RG analysis performed in [20]. Accordingly, various conclusions concerning the physical behaviour of the RF Hamiltonian, most notably about its localization behaviour, can be immediately inferred.

In particular, in [20] it was shown that, at the band centre, the conductance of the weakly disordered $2d$ model (which is essentially determined by the coupling constant $c_1$) did not change under one-loop perturbative renormalization. This observation suggests that a non-localized state might exist in the middle of the band. Since the stability of the perturbative RG relies merely on the smallness of the parameters $1/c_1, c_3/c_1 \ll 1$, one can infer that, at least for $N \gg 1$, the non-Abelian RF model exhibits metallic behaviour at the band centre†. For $N = 1 (\rightsquigarrow c_1 = O(1))$, the perturbative RG is no longer stabilized by a small parameter and it is impossible to rule out the possibility that strong field fluctuations qualitatively change the band centre behaviour.

It was also predicted in [20] that the $2d$ DoS diverges upon approaching the middle of the band. In order to understand on which energy scale that divergence sets in, and how it will eventually be cut off deep within the random matrix regime, one would have to superimpose perturbative RG techniques onto a non-perturbative treatment of the low-energy regime, a task that is beyond the scope of the present paper.

Finally we notice that, in an RG sense, finite energies $\hat{\epsilon}$ represent a relevant perturbation. Renormalization of the field theory drives the model away from the chiral band centre limit eventually leading to the standard unitary universality class. This result is consistent with the analysis of [4] where it was shown that continuum fermions (i.e. the analogue of lattice fermions close to the *bottom* of the band) subject to a weak random field map onto a unitary $\sigma$-model.

The tendency of sublattice models to exhibit band centre delocalized behaviour persists even in the (quasi) $1d$ case: It was shown in [5, 21] that $1d$ sublattice models with $N$ even exhibit conventional localization behaviour whilst for $N$ odd a delocalized mode remains in the band centre. This parity effect is closely related to the odd/even phenomenon mentioned above in connection with the mean DoS, and indeed it is the boundary action that is responsible for the quasi $1d$ delocalization phenomenon within the $\sigma$-model formulation [22].

*Lattice QCD and random Dirac fermions.* Besides Gade's model, the field theory (2) has at least two other close relatives: In QCD, (2) has been suggested on phenomenological grounds as relevant for the determination of the low-energy spectrum of the Dirac operator [23]. In that context, the base manifold is (4 + 1)-dimensional whilst the fields $T \in U(n_f + 1|1)$, where $n_f$ is the number of quark flavours. For a comprehensive discussion of the QCD-analogue of (2), its connection with ChRMT and its relevance for lattice QCD analyses, we refer the reader to [24]. The similarity between the theories is again a manifestation of the universality of chiral $\sigma$-models or, more physically, the universal consequences chiral symmetries have for

† It is interesting to note that, according to the connections summarized above, the unusual localization properties of the zero energy states of the RF model characterize those of *all* eigenstates of a stochastic non-Hermitian operator.

the long-range properties of random systems. (In QCD, 'randomness' is represented by gauge field fluctuations in the Yang–Mills Hamiltonian.)

Finally, the group structure of the field manifold admits the existence of a Wess–Zumino–Novikov–Witten (WZNW) term,

$$-c_4 \int_0^1 \mathrm{d}\, u \epsilon^{\zeta\mu\nu} \mathrm{str}(\tilde{T}\partial_\zeta \tilde{T}^{-1}\tilde{T}\partial_\mu \tilde{T}^{-1}\tilde{T}\partial_\nu \tilde{T}^{-1})$$

where $\tilde{T}(u=0) = \mathbb{1}$ and $\tilde{T}(u=1) = T$. Within the framework of non-Abelian bosonization, such terms have been found to appear in the long-range modelling of systems with a Dirac-type dispersion in the clean excitation spectrum [25]. However, the actual realization of a WZNW operator in a model with an underlying clean Dirac structure crucially depends on its behaviour under discrete symmetry operations such as coordinate exchange or reflection. For this reason, contrary to previous speculations [26], such a term can be ruled out of consideration in the sublattice model of dirty d-wave superconductivity [27], as well as the $\pi$-flux model. These systems are invariant under under an exchange of the coordinates and, therefore, do not admit for the presence of a WZNW operator.

Summarizing we have shown that the RF model, (i) has ChRMT as a universal low-energy limit, (ii) is predicted to possess a band centre delocalized phase [20] under conditions where a perturbative RG scheme is applicable, (iii) for $\epsilon \neq 0$ renormalizes towards a conventional unitary fixed point model [4], and (iv) bears quantitative similarity to other chirally symmetric model systems.

Having reviewed these features, we finally outline how the field theory (2) of the RF model is obtained from (1). That, in this letter, the derivation is not formulated in more detail is motivated by the observation that not only the degrees of freedom but also the structure of the field theory is, to a large extent, dictated by aspects of symmetry: By analogy to the situation for the 'conventional' supersymmetric $\sigma$-models [18], there are only a few $GL(n|n)$ invariant operators with $\leqslant 2$ gradients (namely the ones appearing in (2) plus the WZNW operator†). Thus, the 'only' job that is left for a microscopic derivation is to decide whether the operators permitted by symmetry are actually realized in the field theory, and to fix their coupling constants. Here we restrict ourselves to a brief outline of that analysis. Details of the calculation will be presented in a separate publication.

(i) As usual in the construction of field theories of disordered problems, we first represent Green functions of the problem in terms of a Gaussian integral over a field $\psi$. Choosing supersymmetry as a way to normalize the resulting functional integrals to unity, the first step exactly parallels the constructions reviewed in [18]. (ii) Next we average over the set $\{U_{ij}\}$. At that stage significant deviations from the standard treatment of *Hermitian* disordered operators occur. A method of exactly averaging over (extended) models involving *unitary* stochasticity has been introduced in [28, 29] and christened the 'colour–flavour transformation'. Following that reference we eliminate the disorder at the expense of introducing a pair of auxiliary fields $\{(Z_{ij}, \tilde{Z}_{ij})\}$ (which play a role analogous to the Hubbard–Stratonovich field $Q$ commonly employed in $\sigma$-model constructions). (iii) Integrating out the $\psi$ we are left with the action

$$S[Z, \tilde{Z}] = -N \sum_{\langle i\in A, j\in B\rangle} \mathrm{str}\ln(1 - Z_{ij}\tilde{Z}_{ij}) + N \sum_{i\in A} \mathrm{str}\ln\left(\hat{\epsilon} + \sum_{j\in \mathbb{N}_i} Z_{ij}\right) + N \sum_{j\in B} \mathrm{str}\ln\left(\hat{\epsilon} + \sum_{i\in \mathbb{N}_j} \tilde{Z}_{ij}\right) \qquad (5)$$

† To complete the list of symmetry-permitted operators one has to add, $O_i^{(1)} \equiv \mathrm{str}(T\partial_i T^{-1})$ and $O_{ij}^{(2)} \equiv \partial_i \mathrm{str}(T\partial_j T^{-1})$. Operators of this structure are *implicitly* contained in (2) via the boundary term; applying Stokes theorem to $S_\mathrm{b}$ leads to expressions involving $O^{(1)}$ and $O^{(2)}$.

where the notation $j \in \mathbb{N}_i$ indicates that $j$ is summed over all nearest neighbours of $i$. (iv) Subjecting (5) to a saddle-point analysis, the fields are conveniently parametrized as $(Z, \tilde{Z}) \equiv (\mathrm{i}xPT, \mathrm{i}xT^{-1}P)$, where $x$ is a constant, and $T, P \in GL(n|n)$ respectively have the significance of Goldstone, massive modes of the theory. (v) Integrating out $P$ we find that, in contrast to standard $\sigma$-model analyses (and in accord with the construction of Gade's action [20]), a residual coupling between massive and Goldstone modes exists; it gives rise to the $c_3$-term in (2). (vi) The remaining, pure Goldstone action is subjected to a gradient expansion which results in (2).

Summarizing, we have derived an effective field theory for the maximally disordered RF model. The theory has a status analogous to the supersymmetric non-linear $\sigma$-models of 'conventional' disordered Fermi systems, but its behaviour is substantially different, a fact that is readily traced back to the presence of a chiral symmetry. It was shown that the formalism provides a platform from which interconnections to a variety of other recently investigated chiral problems can be conveniently analysed.

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